Multi-unit Double Auction under Group Buying[1]

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Abstract. Group buying is a business model in which a number of buyers join together to make an order of a product in a certain quantity in order to gain a desirable discounted price. Such a business model has recently received significant attention from researchers in economics and computer science, mostly due to its successful application in online businesses, such as *Groupon*[5]. This paper deals with the market situation when multiple sellers sell a product to a number of buyers with discount for group buying. We model this problem as a multi-unit double auction. We first examine two deterministic mechanisms that are budget balanced, individually rational and only one-sided truthful, i.e. it is truthful for either buyers or sellers. Then we find that, although there exists a "trivial" (non-deterministic) mechanism that is (weakly) budget balanced, individually rational and truthful for both buyers and sellers, such a mechanism is not achievable if we further require that both the trading size and the payment are neither seller-independent nor buyer-independent. In addition, we show that there is no budget balanced, individually rational and truthful mechanism that can also guarantee a reasonable trading size.

1 Introduction

Group buying (or collective buying power) is when a group of consumers come together and use the old rule of thumb, there is power in numbers, to leverage group size in exchange for discounts. Led by *Groupon*, the landscape for group buying platforms has been growing tremendously during last few years. Due to the advent of social networks, e.g. *facebook*, this simple business concept has been leveraged successfully by many internet companies. Taking the most successful group buying platform *Groupon* for example, a group buying deal is carried out in the following steps:

1. the company searches good services and products (locally) that normally are not well-known to (local) consumers,
2. the company negotiates with a target merchant for a discounted price for their services and the minimum number of consumers required to buy their services in order to get this discount,
3. the company promotes the merchant's services with the discounted price within a period, say two days,
4. if the number of consumers willing to buy the services reaches the minimum during that period, then all the consumers will receive the services with the discounted price, and the company and the merchant will share the revenue. Otherwise, no deal and no loss for any party, especially the merchant and consumers.

All participants benefit from successful group buying deals: consumers enjoy good services with lower prices, merchants promote their services and most likely more consumers will buy their services with normal prices in the future (i.e. group buying also plays a role of advertising), and the company providing the platform benefit from merchants' revenue.

Besides its simple concept and its successful business applications, group buying is not well studied in academia [1, 3, 2, 5]. It is not because the idea is new, but the combination of collective buying power and advertising challenges theoretical analysis. In this work, we extend the simple concept, used by *Groupon* and most other similar platforms, to allow merchants (or sellers) and consumers (or buyers) to express more of their private information (aka *type*). More specifically, instead of one single discounted price for selling a certain number of units of a product, sellers will be able to express different prices for selling different amounts of the product. Buyers will be able to directly reveal the amount they are willing to pay for a product, other than just show interest in buying a product coming with a fixed price. To that end, we do not just enhance the expression of traders' private information, but also reduce the number of no-deal failures that happen when the number of buyers willing to purchase a product does not reach the predetermined minimum on the *Groupon* platform. Moreover, we will allow multiple sellers to build competition for selling identical products.

Given the above extension, what we get is a multi-unit double auction, where there are multiple sellers and multiple buyers exchanging one commodity and each trader (seller or buyer) supplies or demands multiple units of a commodity. Different from the multi-unit double auctions studied previously [7, 4], the focus of this model is group buying and we assume that sellers have unlimited supply and a seller's average unit price is decreasing (non-increasing) when the number of units sold is increasing. The unlimited supply assumption simplifies the utility definition of sellers, and it is not clear to us how to properly define sellers' utility when their supply is limited.

Due to revelation principle, we only consider mechanisms where traders are required to directly report their types. We will propose/examine some mechanisms in terms of, especially, *budget balance*, *individual rationality*, and *truthfulness*, which are three important criteria we usually try to achieve in designing a double auction. Budget balance guarantees that the market owner running the auction does not lose money. Individual rationality incentivises traders to participant in the auction, as they will never get negative utility/benefit for participating in the auction. Truthfulness makes the game much easier for traders to play, because the best strategy can be easily computed for each trader, which is just his true type. Truthfulness also plays an important role for achieving other properties based on traders' truthful types, e.g. *efficiency* (i.e. social welfare maximisation). We will not measure social welfare in this model, due to unlimited supply. However, we will consider the number of

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units exchanged, called *trading size*, which is part of *market liquidity*, indicating the success of an exchange market.

We find that, even without considering other criteria, budget balance, individual rationality and truthfulness are hard to be satisfied together in this model. It is shown that there is no budget balanced, individually rational and truthful auction, given that both the trading size and the payment are neither seller-independent nor buyer-independent, although we do get mechanisms that are budget balanced, individually rational and one-sided truthful, i.e. truthful for either buyers or sellers. We say a parameter of a mechanism is seller-independent (buyer-independent) if its value does not depend on sellers' (buyers') type reports. However, if we allow either the trading size or the payment to be seller-independent or buyer-independent, we will be able to design auctions satisfying budget balance, individual rationality and truthfulness at the same time. In addition, we prove that there is no budget balanced, individually rational and truthful mechanism that can also guarantee trading size.

This paper is organised as follows. After a brief introduction of the model in Section 2, we propose two budget balanced, individually rational and partially truthful (deterministic) mechanisms in Section 3 and 4. Following that, we further check the existence of (weakly) budget balanced, individually rational and truthful mechanisms in Section 5. Finally, we conclude in Section 6 with related and future work.

2 The Model

We study a multi-unit double auction where multiple sellers and multiple buyers exchange one commodity. Each seller supplies an unlimited number of units of a commodity and each buyer requires a certain number of units of the commodity. Each **trader** (seller or buyer) i has a privately observed valuation function (aka **type**) $v_i : \mathbb{Z}^+ \to \mathbb{R}^+$ where the input of the function is the number of units of the commodity and the output is the valuation for those units together.

We assume that sellers' valuation is **monotonic**: $v_i(k) \leq v_i(k+1)$, and satisfies **group buying discount**: $\frac{v_i(k)}{k} \geq \frac{v_i(k+1)}{k+1}$. That is, a seller's valuation is non-decreasing as the number of units to sell increases, while the mean unit valuation is non-increasing (so buyers can get a discount if the mean valuation is decreasing). One intuition for group buying discount constraint is that the average unit production cost may decrease when many units can be produced at the same time. For a buyer i of type v_i requiring $c_i > 0$ units, v_i satisfies $v_i(k) = 0$ for all $k < c_i$ and $v_i(k) = v_i(c_i) > 0$ for all $k \geq c_i$. The first constraint of buyers' valuation says that their demands cannot be partially satisfied. The second assumption says that there is no cost for buyers to deal with extra units allocated to them (*free disposal*). Following [7, 4], we assume that c_i of buyer i is common knowledge. Without loss of generality, we will assume that $c_i = 1$ for each buyer i to simplify the rest of the analysis, and the results under this assumption can be easily extended for general case.

For participating in an auction, each trader is required to report some information (often related to his type) to the auctioneer (i.e. the market owner). Because of the revelation principle [8], we will focus on auctions that require traders to directly report their types. However, traders do not necessarily report their true types.

Let S be the set of all sellers, B be the set of all buyers, and $T = S \cup B$. We assume that $S \cap B = \emptyset$. Let $v = (v_i)_{i \in T}$ denote the type profile of all traders. Let $v_{-i} = (v_1, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$ be the type profile of all traders except trader i. Given trader i of type v_i, we refer to $R(v_i)$ as the set of all possible type reports of i. Similarly, let $R(v)$ be the set of all possible type profile reports of traders with type profile v. We will use $v^B = (v_i)_{i \in B}$ to denote the type profile of buyers, and $v^S = (v_i)_{i \in S}$ for sellers.

Definition 1. *An* ***multi-unit double auction*** *(MDA)* $\mathcal{M} = (\pi, x)$ *consists of an* ***allocation policy*** $\pi = (\pi_i)_{i \in T}$ *and a* ***payment policy*** $x = (x_i)_{i \in T}$*, where, given traders' type profile report* v, $\pi_i(v) \in \mathbb{Z}^+$ *indicates the number of units that seller (buyer)* i *sells (receives), and* $x_i(v) \in \mathbb{R}^+$ *determines the payment paid to or received by trader* i.

Note that the above definition of MDA contains only deterministic MDAs, i.e. given a type profile report, the allocation and payment outcomes are deterministic. We will also consider non-deterministic/random MDAs where the outcomes are random variables. A non-deterministic MDA can be described as a probability distribution over deterministic MDAs.

Given MDA $\mathcal{M} = (\pi, x)$ and type profile v, we say trader i **wins** if $\pi_i(v) > 0$, **loses** otherwise. An allocation π is **feasible** if $\sum_{i \in B} \pi_i(v) = \sum_{i \in S} \pi_i(v)$ and for all S, B and v. An MDA $\mathcal{M} = (\pi, x)$ is feasible if π is feasible. A non-deterministic MDA is feasible if it can be described as a probability distribution over feasible deterministic MDAs. Feasibility guarantees that the auctioneer never takes a short or long position in the commodity exchanged in the market. For the rest, only feasible MDAs are discussed.

Given traders' type profile v, their type profile report $\hat{v} \in R(v)$ and deterministic MDA $\mathcal{M} = (\pi, x)$, the **utility** of trader i with type v_i is defined as

$$u(v_i, \hat{v}, (\pi, x)) = \begin{cases} v_i(\pi_i(\hat{v})) - x_i(\hat{v}), & \text{if } i \in B. \\ x_i(\hat{v}) - v_i(\pi_i(\hat{v})), & \text{if } i \in S. \end{cases}$$

Considering $\mathcal{M}$ might be non-deterministic, we use $E[u(v_i, \hat{v}, (\pi, x))]$ to denote the expected utility of trader i.

Definition 2. *An MDA* $\mathcal{M} = (\pi, x)$ *is* ***truthful*** *(or* ***incentive-compatible****) if* $E[u(v_i, (v_i, \hat{v}_{-i}), (\pi, x))] \geq E[u(v_i, \hat{v}, (\pi, x))]$ *for all* $i \in T$*, all* $\hat{v} \in R(v)$*, all* v.

In other words, a mechanism is truthful if reporting type truthfully maximises each trader's utility. We say an MDA $\mathcal{M}$ is **buyer-truthful** (**seller-truthful**) if $\mathcal{M}$ is truthful for at least buyers (sellers).

An MDA is **budget balanced** (BB) if the payment received from buyers is equal to the payment paid to sellers, and it is **weakly budget balanced** (WBB) if the payment received from buyers is greater than the payment paid to sellers. An MDA is **individually rational** (IR) if it gives its participants non-negative utility. Because of unlimited supply, we will not be able to measure social welfare in this model, as it will be infinite before and after the auction. *Market liquidity*, as an important indicator of a successful exchange market, will be considered. We will check one of the important measures of market liquidity, the number of units exchanged, called **trading size**.

Given type profile report v, assume that $v_1^B(1) \geq v_2^B(1) \geq \cdots \geq v_m^B(1)$, we define the **optimal trading size** $k_{opt}(v)$ as

$$k_{opt}(v) = \max_k \left(\sum_{i=1}^{k} v_i^B(1) \geq \min v_j^S(k)\right). \tag{1}$$

That is, optimal trading size is the maximal number of units that can be exchanged in a (weakly) budget balanced auction, given that the payment of a winning trader is his valuation for receiving/selling the number of units allocated to him. As we will see, it is often not possible to achieve the optimal trading size, if we consider other properties at the same time. Therefore, we define the following notion to measure an MDA's trading size, and similar notions are widely used for analysing online algorithms/mechanisms [9].

Definition 3. *An MDA $\mathcal{M}$ is **c-competitive** if the (expected) trading size $k_{\mathcal{M}}(v)$ of $\mathcal{M}$ is at least $\frac{k_{opt}(v)}{c}$, for all type profile report v. We say $\mathcal{M}$ is **competitive** if $\mathcal{M}$ is c-competitive for a constant c. We refer to c as competitive ratio.*

Moreover, other than following Definition 2, we will use Proposition 1 to analyse the truthfulness of an MDA. Proposition 1 is based on Proposition 9.27 of [9], and its proof directly follows the proof there.

Proposition 1 (Proposition 9.27 of [9])**.** *An MDA $\mathcal{M} = (\pi, x)$ is truthful if and only if it satisfies the following conditions for every trader i with type v_i and every v_{-i}*

- *If $E[\pi_i(v_i, v_{-i})] = E[\pi_i(\hat{v}_i, v_{-i})]$, then $E[x_i(v_i, v_{-i})] = E[x_i(\hat{v}_i, v_{-i})]$. That is, the payment of i does not depend on v_i, but only on the alternative allocation result.*
- *$E[u(v_i, v, (\pi, x))] \geq E[u(v_i, (\hat{v}_i, v_{-i}), (\pi, x))]$ for all $\hat{v}_i \in R(v_i)$. That is, the expected utility of i is optimised by $\mathcal{M}$.*

3 A BB, IR and Buyer-truthful MDA

A Vickrey auction is a truthful and individually rational one-sided auction for exchange of one item, where traders report their private types (valuations for the item), and in which the trader with the highest valuation wins, but the price paid is the second-highest valuation. We apply a similar principle in this section and propose an MDA, called Second Price MDA. We show that this auction is budget balanced and individually rational but only buyer-truthful, i.e. it is truthful for buyers only.

Second Price MDA $\mathcal{M}_{2nd}$

Given type profile report $v = (v^B, v^S)$, assume that $v_1^B(1) \geq v_2^B(1) \geq \cdots \geq v_m^B(1)$.

1. Let $w(k) = \min \operatorname{argmin}_i v_i^S(k)$ and $p(k) = \min_{i \neq w(k)} \frac{v_i^S(k)}{k}$ or ∞ if there is only one seller.
2. Let $k^* = \max\{k | v_k^B(1) \geq p(k)\}$.
3. The first k^* buyers, i.e. buyers of valuation $v_1^B, v_2^B, \cdots, v_{k^*}^B$, receive one unit of the commodity each and each of them pays $p(k^*)$.
4. Seller $w(k^*)$ sells k^* units of the commodity and receives payment $p(k^*) \cdot k^*$.
5. The remaining traders lose without payment.

Given the number of units going to be exchanged k, $\mathcal{M}_{2nd}$ selects the seller with lowest valuation for selling k units to win (i.e. $w(k)$) and the payment is the second lowest valuation (i.e. $p(k) \cdot k$). k^* of $\mathcal{M}_{2nd}$, the trading size, is the maximal number of units that can be exchanged, given that each winning buyer pays the mean unit price $p(k^*)$. It is evident that the profit of the auctioneer running $\mathcal{M}_{2nd}$ will be zero and no participant will get negative utility, i.e. $\mathcal{M}_{2nd}$ is budget balanced and individually rational.

Lemma 1. *For any $k \geq 1$, $p(k)$ of $\mathcal{M}_{2nd}$ satisfies $p(k+1) \leq p(k)$ and $p(k+1) \cdot (k+1) \geq p(k) \cdot k$.*

Proof. Since sellers' valuation satisfies group buying discount, i.e. $\frac{v_i^S(k+1)}{k+1} \leq \frac{v_i^S(k)}{k}$, we get $p(k+1) = \min_{i \neq w(k+1)} \frac{v_i^S(k+1)}{k+1} \leq \min_{i \neq w(k)} \frac{v_i^S(k)}{k} = p(k)$. In other words, the mean unit price is non-increasing as the number of units sold together increases.

Because of $v_i(k+1) \geq v_i(k)$ for each seller i, we conclude $p(k+1) \cdot (k+1) = \min_{i \neq w(k+1)} v_i^S(k+1) \geq \min_{i \neq w(k)} v_i^S(k) = p(k) \cdot k$. □

Theorem 1. *$\mathcal{M}_{2nd}$ is buyer-truthful.*

Proof. The auction result of $\mathcal{M}_{2nd}$ for buyer i is either receiving one unit with certain payment or receiving nothing with no payment. If i received one unit, then $v_i^B(1) \geq p(k^*)$ and the payment of i is $p(k^*)$ which is independent of $v_i^B(1)$. Otherwise, we know that $v_i^B(1) < p(k^*)$ and the payment is zero for i. Therefore, the first property of Lemma 1 is satisfied for all buyers.

In order to prove truthfulness, we need to show that the utility of each buyer is maximised, i.e. the payment is minimised, by $\mathcal{M}_{2nd}$. For all buyers who received a unit, the payment $p(k^*)$ is the same for all of them. If any of the winning buyers with valuation $v_i^B(1)$ reported $\hat{v}_i^B(1) < p(k^*) \leq v_i^B(1)$, this buyer will not win. Moreover, from Lemma 1, we know that $p(k^*)$ is minimal as k^* is maximal. Therefore, $p(k^*)$ is the minimum valuation for buyers to win in $\mathcal{M}_{2nd}$. Thus, the payment $p(k^*)$ for all winning buyers is minimised. This also holds for losing buyers. □

Theorem 2. *$\mathcal{M}_{2nd}$ is not seller-truthful.*

Proof. The auction result of $\mathcal{M}_{2nd}$ for seller i is either selling k units with payment $p(k)$ for some $k > 0$ or selling nothing with no payment. For each $k > 0$, if seller i successfully sells k units, then the payment $p(k) \cdot k$ received by i is the second lowest valuation of sellers for selling k units together and is independent of i's type. If seller i loses, the payment is zero for i. Therefore, the first property of Lemma 1 is also satisfied for all sellers.

The reason why $\mathcal{M}_{2nd}$ is not truthful for sellers is that the utilities of sellers might not be maximised. For instance, assume that k_1 and $k_1 - 1$ satisfy the condition $v_k^B(1) \geq p(k)$, and $w(k_1) = w(k_1 - 1) = i$. If $p(k_1) \cdot k_1 - v_i^S(k_1) < p(k_1 - 1) \cdot (k_1 - 1) - v_i^S(k_1 - 1)$, then i would prefer selling $k_1 - 1$ units other than k_1 units. Therefore, if i sells k_1 units with payment $p(k_1) \cdot k_1$, she is incentivised to manipulate the auction in order to sell only $k_1 - 1$ units with more utility. The manipulation will be successful if the third lowest seller valuation for selling k_1 units, say $v_j^S(k_1)$, satisfies $\frac{v_j^S(k_1)}{k_1} > v_{k_1}^B(1)$ (by simply misreporting $\hat{v}_i^S(k_1) \geq v_{k_1}^B(1)$). □

4 A BB, IR and Seller-truthful MDA

In the last section, we showed that a simple second price MDA is not truthful, because sellers' utilities are not maximised. However, in this section, we will see that if we simply update $\mathcal{M}_{2nd}$ such that sellers' utilities are maximised, but then buyers will sacrifice. The main update is that the determination of the trading size considers the winning seller's utility.

Second Price plus Seller Utility Maximisation MDA $\mathcal{M}_{2nd}^+$

Given type profile report $v = (v^B, v^S)$, assume that $v_1^B(1) \geq v_2^B(1) \geq \cdots \geq v_m^B(1)$.

1. Let $w(k) = \min \operatorname{argmin}_i v_i^S(k)$ and $p(k) = \min_{i \neq w(k)} \frac{v_i^S(k)}{k}$ or ∞ if there is only one seller.

2. Let $k^* = \max\{k | v_k^B(1) \geq p(k)\}$, and $i^* = w(k^*)$.
3. Let $K = \{k | v_k^B(1) \geq p(k)\}$, and K^* is the least set such that $i^* \in K^*$ and $K^* \supseteq \{k | k = \max(K \setminus K^*) \wedge w(k) = i^* \wedge v_{\min K^*}^B(1) < \frac{v_{3rd}^S(\min K^*)}{\min K^*}\}$, where $v_{3rd}^S(k)$ is the third lowest valuation of sellers for selling k units and it is ∞ if there are less than three sellers.
4. Let $k_+^* = \max \operatorname{argmax}_{k \in K^*}(p(k) \cdot k - v_{i^*}^S(k))$.
5. The first k_+^* buyers, i.e. buyers of valuation $v_1^B, v_2^B, \cdots, v_{k_+^*}^B$, receive one unit of the commodity each and each of them pays $p(k_+^*)$.
6. Seller i^* sells k_+^* units of the commodity and receives payment $p(k_+^*) \cdot k_+^*$.
7. The rest of the traders lose without payment.

k^* and the winning seller i^* of $\mathcal{M}_{2nd}^+$ is the same as that in $\mathcal{M}_{2nd}$. Set K contains all possible numbers of units that can be exchanged without sacrificing budget balance. Set K^* contains all k points that seller i^* can manipulate and force the auctioneer to choose some $k^* \in K^*$ if $\mathcal{M}_{2nd}$ is used. The reason is that, for all $k \in K^*$ except the minimum ($\min K^*$), seller i^* is the only winner, i.e. without seller i^*, there is no other seller who can win at those points. Therefore, $\mathcal{M}_{2nd}^+$ chooses $k_+^* \in K^*$, as the final trading size, such that seller i^*'s utility is maximised among all $k \in K^*$. It is evident that $\mathcal{M}_{2nd}^+$ is also budget balanced and individually rational.

Theorem 3. *$\mathcal{M}_{2nd}^+$ is seller-truthful but not buyer-truthful.*

Proof. Regarding truthfulness of sellers, firstly, their payments are independent of their valuations. Secondly, their utilities are maximised, i.e. they cannot misreport their valuations to get higher utilities. For winning seller i^*, K^* contains all winning k points where i^* is the winner and she can manipulate to get a winning point giving her the highest utility. However, seller i^* cannot misreport to win at other winning points outside of K^*. This is because another seller will win at either $\min K^*$ or $\max(K \setminus K^*)$ if seller i^* chooses to not win at any point in K^*. Since $\mathcal{M}_{2nd}^+$ selects the winning point $k_+^* \in K^*$ that gives i^* the highest utility she could possibly get with misreporting, there is no reason for i^* to misreport. For a losing seller i, if i misreported and won at k^*, then i has to misreport $\hat{v}_i^S(k^*) \leq v_{i^*}^S(k^*) \leq v_i^S(k^*)$ and the K^* for i will be $\{i^*\}$. Therefore, i will get non-positive utility, $v_{i^*}^S(k^*) - v_i^S(k^*)$, in order to win at point k^*. If i misreported and won at a point $k' > k^*$, then i has to misreport $\hat{v}_i^S(k') \leq v_{i'}^B(1) \cdot k' \leq v_i^S(k')$ and the new unit price $\hat{p}(k')$ must satisfy that $\frac{\hat{v}_i^S(k')}{k'} \leq \hat{p}(k') \leq v_{i'}^B(1)$. Thus the utility for losing seller i to win at point k' will be $\hat{p}(k') \cdot k' - v_i^S(k') \leq 0$. Therefore, truthfulness also holds for losing sellers.

It is evident that $\mathcal{M}_{2nd}^+$ is not truthful for buyers because their payments $p(k_+^*) \geq p(k^*)$ (Lemma 1). That is, buyers of valuation $v_1^B, v_2^B, \cdots, v_{k^*}^B$ could misreport their valuations to prevent seller i^* winning at any point $k_+^* < k^*$, which might give them higher utilities. □

Proposition 2. *The utility loss of winning buyer i in $\mathcal{M}_{2nd}^+$, compared with the utility i can achieve in $\mathcal{M}_{2nd}$, is not more than $\frac{k^* - k_+^*}{k_+^*}$ of the payment i can get when i participates in $\mathcal{M}_{2nd}$.*

Proof. According to Lemma 1, we get $p(k^*) \cdot k^* \geq p(k_+^*) \cdot (k_+^*)$. Therefore, for a winning buyer i of type v_i in $\mathcal{M}_{2nd}^+$, i's utility $u_{\mathcal{M}_{2nd}^+} = v_i(1) - p(k_+^*)$, while the utility i will get in $\mathcal{M}_{2nd}$ is $u_{\mathcal{M}_{2nd}} = v_i(1) - p(k^*)$. So we get $u_{\mathcal{M}_{2nd}} - u_{\mathcal{M}_{2nd}^+} = p(k_+^*) - p(k^*) \leq \frac{k^* - k_+^*}{k_+^*} p(k^*)$. □

5 Existence of (W)BB, IR and Truthful MDAs

Following the results in previous sections, we demonstrate in this section that there are multi-unit double auctions that are (weakly) budget balanced, individually rational and truthful. However, we also prove that there does not exist a (weakly) budget balanced, individually rational and truthful MDA, in which both the trading size and the payment are neither seller-independent nor buyer-independent.

Proposition 3. *There exists (weakly) budget balanced, individually rational, and truthful multi-unit double auctions.*

Proof. The fixed pricing MDA described in Auction 1 is BB, IR and truthful. Given a predetermined transaction price p, $\mathcal{M}_{fixed}$ first calculates the total number k_1 of buyers whose valuations are at least p, then calculates the maximal number k^* of units that a seller can sell, with non-negative utility, under unit price p, given that $k^* \leq k_1$. After it calculates all the winning candidates of both sides, candidates from the same side win with the same probability. It is evident that this auction is budget balanced and individually rational.

Regarding truthfulness, firstly, payment p does not depend on any trader. Secondly, all buyers whose valuation for one unit is at least p will win with the same probability with payment p, so their utilities are maximised if their winning probability $\frac{k^*}{k_1}$ is maximised. Buyer i of $v_i^B(1) \geq p$ will not report $\hat{v}_i^B(1) < p$ as i's winning probability will be reduced. Also buyer i of $v_i^B(1) < p$ will not report $\hat{v}_i^B(1) \geq p$ because he will get a negative expected utility. Therefore, k_1 is fixed for a given type profile report and no buyer is incentivsed to change it. Moreover, k^* is maximised. Thus, $\frac{k^*}{k_1}$ is maximised and buyers' utilities are maximised. A similar analysis applies to sellers. □

Auction 1 (Fixed Pricing MDA $\mathcal{M}_{fixed}$). *Given predetermined transaction price p and type profile report $v = (v^B, v^S)$,*

1. *let $k_1 = |\{i | v_i^B(1) \geq p\}|$,*
2. *let $k^* = \max\{k | k \leq k_1 \wedge \frac{v_i^S(k)}{k} \leq p$ for some $i\}$, and $k_2 = |\{i | \frac{v_i^S(k^*)}{k^*} \leq p\}|$,*
3. *randomly select k^* winning buyers from $\{i | v_i^B(1) \geq p\}$, i.e. each buyer $i \in \{i | v_i^B(1) \geq p\}$ wins with probability $\frac{k^*}{k_1}$,*
4. *randomly choose one winning seller from $\{i | \frac{v_i^S(k^*)}{k^*} \leq p\}$, i.e. each seller $i \in \{i | \frac{v_i^S(k^*)}{k^*} \leq p\}$ wins with probability $\frac{1}{k_2}$,*
5. *each winning buyer receives one unit of the commodity and pays p, the winning seller sells k^* units and receives payment $p * k^*$, and the remaining traders lose with no payment.*

Note that $\mathcal{M}_{fixed}$ is non-deterministic and the payment p does not depend on any trader. It is not hard to check that similar auctions with two fixed prices p_s, p_b such that $p_s \leq p_b$ and p_s is the unit price for winning sellers and p_b for winning buyers is (W)BB, IR and truthful. Other than fixed pricing MDAs, there are (W)BB, IR and truthful MDAs where payments are not predetermined. For instance, a simple variant of $\mathcal{M}_{fixed}$ described in Auction 2 is one such mechanism and it is clear that $\mathcal{M}_{single}$ is BB, IR and truthful. However, there is no MDA that is (W)BB, IR and truthful, given that both the trading size and the payment are neither seller-independent nor buyer-independent. We say a parameter of an MDA is seller-independent (buyer-independent) if the value of the parameter does not depend on sellers' (buyers') type reports.

Definition 4. *Given MDA $\mathcal{M}$, a parameter d of $\mathcal{M}$, and type profile $v = (v^B, v^S)$, we say d is **trader-independent** if the value of d, denoted by $d_{\mathcal{M}}(\cdot)$, satisfies $d_{\mathcal{M}}(\hat{v}) = d_{\mathcal{M}}(\bar{v})$ for all $\hat{v}, \bar{v} \in R(v)$. We say d is **seller-independent** if $d_{\mathcal{M}}((\hat{v}^B, \hat{v}^S)) = d_{\mathcal{M}}((\hat{v}^B, \bar{v}^S))$ for all $\hat{v}^B \in R(v^B)$, all $\hat{v}^S, \bar{v}^S \in R(v^S)$. We say d is **buyer-independent** if $d_{\mathcal{M}}((\hat{v}^B, \hat{v}^S)) = d_{\mathcal{M}}((\bar{v}^B, \hat{v}^S))$ for all $\hat{v}^B, \bar{v}^B \in R(v^B)$, all $\hat{v}^S \in R(v^S)$.*

A parameter of an MDA is trader-independent if and only if it is seller-independent and buyer-independent. For instance, p of $\mathcal{M}_{fixed}$ is trader-independent, and p of $\mathcal{M}_{single}$ is seller-independent.

Auction 2 (One-sided Pricing MDA $\mathcal{M}_{single}$)**.** *Given type profile report $v = (v^B, v^S)$,*

1. *let p be the $\lceil \frac{m}{2} \rceil$-th highest of $v_i^B(1)$s, where m is the total number of buyers,*
2. *let $k_1 = |\{i | v_i^B(1) > p\}|$,*
3. *let $k^* = \max\{k | k \leq k_1 \wedge \frac{v_i^S(k)}{k} \leq p$ for some $i\}$, and $k_2 = |\{i | \frac{v_i^S(k^*)}{k^*} \leq p\}|$,*
4. *randomly select k^* winning buyers from $\{i | v_i^B(1) > p\}$, i.e. each buyer $i \in \{i | v_i^B(1) > p\}$ wins with probability $\frac{k^*}{k_1}$,*
5. *randomly choose one winning seller from $\{i | \frac{v_i^S(k^*)}{k^*} \leq p\}$, i.e. each seller $i \in \{i | \frac{v_i^S(k^*)}{k^*} \leq p\}$ wins with probability $\frac{1}{k_2}$,*
6. *each winning buyer receives one unit of the commodity and pays p, the winning seller sells k^* units and receives payment $p * k^*$, and all the rest of the traders lose with no payment.*

Theorem 4. *There is no (weakly) budget balanced, individually rational and truthful multi-unit double auction, where both the trading size and the payment are neither seller-independent nor buyer-independent.*

Before we give the proof of Theorem 4, we first prove some lemmas that are going to be used for the proof. Lemma 2 says that an IR and truthful MDA cannot have price discrimination. An MDA has *price discrimination* if buyers (sellers) pay (receive) different payments for identical goods or services. For instance, when two buyers pay different prices for receiving one unit of the same commodity in a deterministic MDA, this is considered as price discrimination.

Lemma 2. *An individually rational multi-unit double auction with price discrimination is not truthful.*

Proof. Because of individual rationality, the expected payments for all winning buyers (sellers) must not over (under) their valuations.[6] If the expected payments are not the same between winning buyers/sellers, then a winning buyer (seller) with high (low) expected payment will have a chance to manipulate the auction in order to get a low (high) expected payment by, for example, reporting the same valuation as that of a winning buyer (seller) receiving relatively a lower (higher) expected payment. □

[6] Note that we consider expected payment to check price discrimination, because if an MDA is non-deterministic and it can assign different payments to winning buyers/sellers. However, if a non-deterministic MDA is individually rational and truthful, then the expected payment will be the same for all winning buyers/sellers and the prices should be randomly chosen from some range independent of winning traders' valuations. A non-deterministic MDA is not considered price discrimination if the expected payment is the same for all winning/losing buyers/sellers.

From Lemma 2, we conclude that an individually rational and truthful MDA must give the same (expected) payment for all winning buyers/sellers, and give no payment for all losing traders.

Lemma 3. *All winning sellers in a truthful multi-unit double auction sell the same expected number of units.*

Proof. According to Lemma 2, we know that all winning sellers receive the same expected payment for selling each unit. So their utilities will be higher if they sell more units. If the expected number of units to be sold is not the same among winning sellers, then a seller selling less units is incentivised to manipulate the auction in order to sell more units by simply misreporting his valuation as the seller selling relatively more units. □

Proof of Theorem 4. We first assume that there is such MDA $\mathcal{M}$, and then we end up with a contradiction.

Let p_s and p_b be the payment (unit price) for winning sellers and winning buyers respectively. According to Lemma 3, without loss of generality, we assume that $\mathcal{M}$ selects at most one winning seller. Assume the trading size is k. Let v_{min}^B be the minimum valuation (for one unit) of all winning buyers, and v_{max}^B be the maximum valuation of all losing buyers ($v_{max}^B = 0$ if there is no losing buyer). Let v_{win}^S be the valuation of the winning seller for selling k units, and v_{min}^S be the minimum valuation of all losing sellers for selling k units ($v_{min}^S = \infty$ if there is no losing seller). Because of individual rationality, we have $\frac{v_{win}^S}{k} \leq p_s \leq p_b \leq v_{min}^B$. Since $\mathcal{M}$ is truthful, we further get $p_s \leq \frac{v_{min}^S}{k}$ and $p_b \geq v_{max}^B$ and p_s and p_b should not depend on any winning trader. Therefore, if $\mathcal{M}$ chooses any k satisfying any of the following four conditions, there will be proper payments $p_s \leq p_b$ only depending on v_{max}^B and v_{min}^S.

1. $\frac{v_{min}^S}{k} \leq v_{max}^B$,
2. $\frac{v_{min}^S}{k} > v_{max}^B$, $v_{min}^B \geq \frac{v_{min}^S}{k}$, and $v_{max}^B \geq \frac{v_{win}^S}{k}$,
3. $\frac{v_{min}^S}{k} > v_{max}^B$, $v_{min}^B \geq \frac{v_{min}^S}{k}$, and $v_{max}^B < \frac{v_{win}^S}{k}$,
4. $\frac{v_{min}^S}{k} > v_{max}^B$, $v_{min}^B < \frac{v_{min}^S}{k}$, and $v_{max}^B \geq \frac{v_{win}^S}{k}$.

For condition (1), $p_b, p_s \in [\frac{v_{min}^S}{k}, v_{max}^B]$ s.t. $p_s \leq p_b$. For condition (2), $p_b, p_s \in [v_{max}^B, \frac{v_{min}^S}{k}]$ s.t. $p_s \leq p_b$. For condition (3), $p_b = p_s = \frac{v_{min}^S}{k}$, and $p_b = p_s = v_{max}^B$ for condition (4).

In other words, $\mathcal{M}$ chooses any k satisfying any of the above four conditions can also get payments independent of winning traders and satisfying (weakly) budget balance. Besides these four conditions, we cannot choose any k under other conditions where we can still get (weakly) budget balanced and winning trader independent payments, given that both k and p_s, p_b are neither seller-independent nor buyer-independent.

Therefore, in order to satisfy truthfulness, M has to choose a k such that all traders' utilities are maximised. For winning buyers, they would prefer a bigger k as their payment will be lower compared to the payment with a lower k, i.e. their utilities are maximised when k is maximised. However, the winning seller might prefer a lower k as her utility is not necessarily maximised with maximum k (see the proof of Theorem 2 for example). Thus, we may not always be able to choose a k maximising both buyers' and sellers' utilities. This contradicts the truthfulness of $\mathcal{M}$, i.e. buyers may be incentivised to disable the above four conditions for lower ks, while sellers may be motivated to disable that for higher ks. □

5.1 Competitive MDAs

Corollary 1. *There is no (weakly) budget balanced, individually rational, truthful multi-unit double auction that is also competitive.*

Proof. From Theorem 4, we know that there is no (W)BB, IR, truthful, and competitive multi-unit double auction, if both the trading size and the payment are neither seller-independent nor buyer-independent. In the following, we will prove that if the trading size or the payment of an MDA is either seller-independent or buyer-independent, the MDA will not be competitive.

If the trading size of MDA $\mathcal{M}$ is seller-independent, say the expected trading size is k_e, then k_e must be also buyer-independent, otherwise we can always find a example that violates budget balance, individual rationality and truthfulness. For instance, each seller's unit valuation for selling any number of units is larger than the highest valuation of sellers, in which the trading size should be zero if BB, IR and truthfulness are satisfied. Therefore, given $k_e > 0$ is trader-independent, for any type profile report v with optimal trading size $k_{opt}(v)$, the competitive ratio $c = \frac{k_{opt}(v)}{k_e}$. It is clear that c is not bounded as $k_{opt}(v)$ can be any value approaching to infinite.

If the payment of MDA $\mathcal{M}$ is seller-independent, then for any payment determined without considering sellers, there exists a case where all sellers' unit valuation for selling any number of units are higher than the payment, which means that the trading size will be zero if $\mathcal{M}$ is (weakly) budget balanced, individually rational, truthful. Therefore, $\mathcal{M}$ cannot be competitive under this condition. This result also holds when the payment is buyer-independent. □

6 Conclusion

In this paper, we studied a multi-unit double auction, where each seller has an unlimited supply, for exchanging one kind of commodity. Different from the previous studies of multi-unit double auction, we introduced group buying in the model. More specifically, sellers' average unit valuation is decreasing (non-increasing) as the number of units sold together increases, i.e. more buyers buying the commodity together as a group from a seller will result in a higher discount.

We found that, under this model, even without considering other criteria, budget balanced, individually rational and truthful mechanisms are hard to achieve. We showed that in Theorem 4 there is no budget balanced, individually rational and truthful multi-unit double auction, if both the trading size and the payment of the auction are neither seller-independent nor buyer-independent, although we got mechanisms in Section 3 and 4 that are budget balanced, individually rational and one-sided truthful, i.e. truthful for either buyers or sellers. However, if we allow either the trading size or the payment to be seller-independent or buyer-independent, in Section 5, we did get auctions that satisfy all the three criteria. Moreover, if we consider trading size (i.e. the number of units exchanged) at the same time, we demonstrated in Corollary 1 that there is no budget balanced, individually rational and truthful mechanism that can also guarantee trading size.

The results in this paper are based on the assumption that each buyer requires only one unit. As we mentioned, the results are applicable to the general case where each buyer i requires $c_i > 0$ units. For the extension, we just need to update $v_i^B(1)$ into $\frac{v_i^B(c_i)}{c_i}$ in the results, and count the number of units for a buyer group based on buyers' c_is other than the number of buyers in the group. For non-deterministic MDAs, e.g. $\mathcal{M}_{fixed}$ and $\mathcal{M}_{single}$, the winning probability of a buyer will be based on his c_i, e.g. the winning probability of buyer i in step 3 of $\mathcal{M}_{fixed}$ will be $\frac{k^* \cdot c_i}{k_1}$. As c_is are not part of buyers' private information, this extension will not affect any of the properties that hold in the single-unit demand case.

As closely related work, Huang *et al.* [7] proposed weakly budget balanced, individually rational and truthful multi-unit double auctions, under the model where each seller (buyer) supplies (demands) a publicly known number of units, their valuation for each unit is not changing and their requirements can be partially satisfied. Chu [4] studied a multi-unit double auction model where there are multiple commodities, each seller supplies multi-units of one commodity and each buyer requires a bundle of different commodities. They proposed a method that intentionally creates additional competition in order to get budget balanced, individually rational and truthful mechanisms. Wurman *et al.* [10] also considered one-sided truthful double auctions for optimising social welfare. Goldberg *et al.* [6] studied one-sided auctions where the seller has an unlimited supply without giving any valuation or reserve price for the commodity, and their gaol is to design truthful mechanisms that guarantee the seller's revenue. For group buying, Edelman *et al.* [5] considered the advertising effect of discount offers by modelling the procedure with two periods, so traders can come back in the future after getting discounted offers. Arabshahi [2] provided a very detailed analysis of the *Groupon* business model and Byers *et al.* [3] showed some primary post-analysis of *Groupon*. A very earlier study of online group buying is provided by Anand and Aron [1].

There are many questions for considering group buying in multi-unit double auction worth further investigation. Especially, if sellers have limited supply, how do we calculate their utilities, as they should have valuation for the unsold units and the valuation for the unsold units is not the same before and after the auction, raising the further question of how to optimise social welfare and guarantee other properties in this case. For instance, a seller supplies two units with unit prices $p_1 > p_2$ for selling one and two units respectively. If we end up with one unit left for the seller, we might consider that the seller has a valuation of p_1 for this unsold unit.

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