CS243: Introduction to Algorithmic Game Theory

Week 3.1, VCG (Dengji ZHAO)

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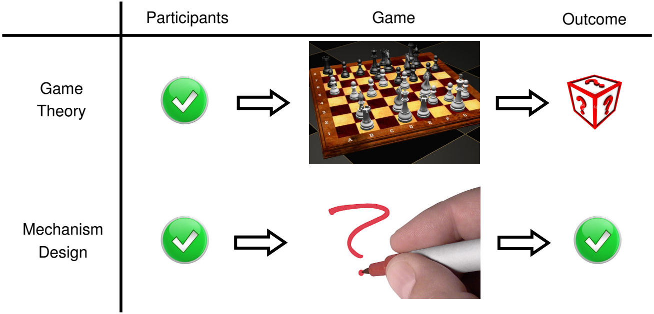
Recap: Game Theory



Recap: The General Setting of Mechanism Design

- A set of n participants/players, denoted by N.
- A mechanism needs to choose some alternative from A (allocation space), and to decide a payment for each player.
- Each player $i \in N$ has a private valuation function $v_i : A \to \mathbb{R}$, let V_i denote all possible valuation functions for i.
- Let $v = (v_1, \cdots, v_n)$, $v_{-i} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$.
- Let $V = V_1 \times \cdots \times V_n$, $V_{-i} = V_1 \times \cdots V_{i-1} \times V_{i+1} \times \cdots \times V_n$.

Recap: A Definition of a Mechanism (with Money)

Definition

A (direct revelation) mechanism is a social choice function $f : V_1 \times \cdots \times V_n \to A$ and a vector of payment functions $p_1, \ldots, p_n$, where $p_i : V_1 \times \cdots \times V_n \to \mathbb{R}$ is the amount that player i pays.

- direct revelation: *the mechanism requires each player to report her valuation function to the mechanism.*

Definition

Given a mechanism $(f, p_1, \ldots, p_n)$, and players' valuation report profile $v' = (v'_1, \cdots, v'_i, v'_n)$, player i's utility is defined by $v_i(f(v')) - p_i(v')$, where v_i is i's true valuation function.

Recap: Properties of a Mechanism

Truthfulness A mechanism $(f, p1, \ldots, p_n)$ is called truthful (*incentive compatible*) if for every player i, every $v_1 \in V_1, \ldots, v_n \in V_n$ and every $v_i' \in V_i$, if we denote $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$, then $v_i(a) - p_i(v_i, v_{-i}) \geq v_i(a') - p_i(v_i', v_{-i})$.

Efficiency We say a social choice function f is efficient if it maximises social welfare for all valuation reports. That is, for all $v \in V$, $f \in \arg\max_{f' \in F} \sum_{i \in N} v_i(f'(v))$ where F is the set of all feasible social choice functions.

Individual Rationality We say a mechanism $(f, p_1, \ldots, p_n)$ is individually rational if for every player i, every $v \in V$, we have $u_i(f, p_1, \ldots, p_n, v, v_i) \geq 0$.

Vickrey-Clarke-Groves Mechanisms

Definition 9.16 A mechanism $(f, p_1, \ldots, p_n)$ is called a Vickrey–Clarke–Groves (VCG) mechanism if

- $f(v_1, \ldots, v_n) \in \operatorname{argmax}_{a \in A} \sum_i v_i(a)$; that is, f maximizes the social welfare, and
- for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \to \Re$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$: $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$.

$+ v_i(f(v_1, \ldots, v_n)) = \sum_i v_i(a)$

Vickrey-Clarke-Groves Mechanisms

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$$u_i(\;) = v_i(a) + \sum_{j \neq i} v_j(a) = \sum_{i \in N} v_i(a)$$

- Definition of $h_{-i} : V_{-i} \to \mathbb{R}$
  - $h_{-i}(.) = 0$
  - $h_{-i}(v_{-i}) = \sum_{j \in N \setminus \{i\}} v_j(f(v_{-i}))$, the maximum social welfare without i's participation.
  - ...

	2nd	$h_i = 0$
v_1	$v_1 - v_2$	$v_1 - 0 = v_1$
v_2	0	$0 - P_2$
v_3	0	$0 + v_1 = v_1$

$$h_i = \sum_{j \neq i} v_j(f(v_{-i}))$$

$$P_1 = v_2 - 0$$

$$P_2 = v_1 - v_1 = 0$$

$$P_3 = 0$$

Examples of Applying VCG

A seller sells m (heterogeneous) items:

- A set of m items to be allocated (denoted by M)
- A set of n players (denoted by N)
- Each player i has a valuation function $v_i : 2^M \to \mathbb{R}$

Question

What is size of the allocation space?

Properties of VCG

Is VCG truthful, efficient and individually rational?

How to verify a mechanism is truthful or not?

Theorem

A mechanism is truthful if and only if it satisfies the following conditions for every i and every v_{-i}:

1. *The payment p_i does not depend on v_i, but only on the alternative chosen $f(v_i, v_{-i})$. That is, for every v_{-i}, there exist prices $p_a \in \mathbb{R}$, for every $a \in A$, such that for all v_i with $f(v_i, v_{-i}) = a$ we have that $p(v_i, v_{-i}) = p_a$.*
2. *The mechanism optimizes for each player. That is, for every v_i, we have that $f(v_i, v_{-i}) \in \arg\max_a (v_i(a) - p_a)$, where the quantification is over all alternatives in the range of $f(\cdot, v_{-i})$.*

* Assume exist v_i' $f(v_i', v_{-i}) = a'$, $u_i' > u_i$

$$u_i = v_i(a^*) - \left(\cancel{h_i(v_{-i})} - \sum_{j \neq i} v_j(a^*)\right) = \sum_i v_i(a^*)$$

$$\wedge \qquad\qquad\qquad\qquad \wedge$$

$$u_i' = v_i(a') - \left(\cancel{h_i(v_{-i})} - \sum_{j \neq i} v_j(a')\right) = \sum_i v_i(a')$$

$$f(v) \in \arg\max_{a \in A} \sum_i v_i(a)$$

$$a^* \Rightarrow \sum_i v_i(a^*) \geq \sum_i v_i(a')$$

IR:

$$u_i = v_i(a^*) - \left(h_i(v_{-i}) - \sum_{j \neq i} v_j(a^*)\right)$$

$$= \sum_i v_i(a^*) - h_i(v_{-i}) \geq 0$$

$$\Rightarrow h_i(v_{-i}) \leq \sum_i v_i(a^*) \qquad a^* = f(v)$$

$$\sum_{j \neq i} v_j(f(v_{-i})) \quad ? \quad + v_i(f(v_{-i})) = \sum_i v_i(a')$$

$$\vee\!|$$

	iPhone 1	iPhone 20	1+20
U_1	200	1000	1100
× U_2	250 ✓	1250 ✓	500
U_3	100	500	550
× U_4	300 ✓	1300 ✓	1500
U_{2+4}	300	1300	1500

h_i

$P_2 = 1500 - 1300 = 200$

$u_2 = 250 - 200 = 50$

$P_4 = 1450 - 250 = 1200$

$u_4 = 100$

$P_{2+4} = 1100 - 0 = 1100$

$u_{2+4} = 400$

	A	B	AB
V_1	10	0	10
V_2	0	10	10
V_3	3	3	8
V_{1+2}	10	10	20

$P_1 = 13 - 10 = 3$ $\quad u_1 = 10 - 3 = 7$

$P_2 = 13 - 10 = 3$ $\quad u_2 = 10 - 3 = 7$

14

$P_{1+2} = 8 - 0 = 8$

$u_{1+2} = 20 - 8 = 12$

false-name manipulation

Advanced Reading

- Introduction to Mechanism Design [AGT Chapter 9]
- Vickrey-Clarke-Groves mechanisms [AGT Chapter 9.3]

$v_i \in [0,1]$ Uniform Random

v_1

v_1, v_2

v_3

v_n

Rev. $= v_2$

Worst Case : Rev $= 0$

Best Case : Rev $= 1$

Avg. Case : $\frac{1}{2}$?

$P[v_2]$ $\begin{cases} P[\leq 1] = 1 \\ P[\geq 0] = 1 \end{cases}$

$* P[\leq x] = 1 - (1-x)^2$

$F(x)$

$* P[\geq x] = (1-x)^2$ $\quad f(x) = 2(1-x)$

$v_1 \geq x \quad v_2 \geq x \quad \cancel{v_3 \leq v_2}$

① $E[Rev] = \int_0^1 x f(x) dx = \int_0^1 2x(1-x)\, dx = \int_0^1 2x - 2x^2 dx$

$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

r : reserve price

$p = \max(r, \min(v_1, v_2))$

$E[Rev] = x^2 - \frac{2}{3}x^3 \Big|_r^1 + P[v_2 < r] \cdot r \cdot P[v_1 \geq r]$

$2 \cdot r \cdot r \cdot (1-r)$

$$= \frac{1}{3} - \left(r^2 - \frac{2}{3}r^3\right) + 2r^2(1-r)$$

$$= \frac{1}{3} + r^2 - \frac{4}{3}r^3$$

$$2r - 4r^2 = 0 \Rightarrow r = 0 \;/\; r = \frac{1}{2}$$

$$r = 0 \downarrow \frac{1}{3} \qquad r = \frac{1}{2} \downarrow \frac{1}{3} + \frac{1}{4} - \frac{\cancel{4}}{3} \cdot \frac{1}{\cancel{8}} = \frac{7}{12} - \frac{2}{12} = \frac{5}{12}$$

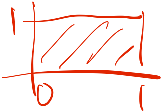
$v_i \in [0, 1]$

$F(v_i)$, $f(v_i)$

$$\boxed{\begin{matrix} v_1 & & v_1' & & & & \\ V/v_2 & \overset{\phi}{\Longrightarrow} & v_2' & \overset{VCG}{\Longrightarrow} & w' & & w = w' \\ \vdots & & \vdots & & p' & \Longrightarrow & p = \phi^{-1}(p') \\ v_n & & v_n' & & & & \end{matrix}}$$

$$\phi(v_i) = \underline{v_i - \frac{1 - F(v_i)}{f(v_i)}} = v_i - \frac{1 - v_i}{1} = \frac{2v_i - 1}{1}$$

$v_i \in [0,1]$ Uniform Rand. $F(v_i) = v_i$ ✓



$f(v_i) = 1$

$\phi^{-1}(x)$

$x = 2v_i - 1$

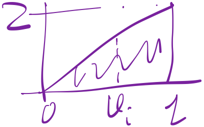
$v_i = \frac{x+1}{2}$

$$\boxed{\begin{array}{lcl} 0.8 & & 0.6 \ \checkmark \ P' = 0 \\ 0.4 & \xrightarrow{\phi} & -0.2 \\ 0.3 & & -0.4 \end{array} \quad w = 1 \quad \Rightarrow P = \phi^{-1}(P') = \frac{1}{2}}$$

$r + VCG$
$\parallel$
$1/2$

$w = 1$

$$\Rightarrow P = \max\left(\tfrac{1}{2}, 0.4\right) = \frac{1}{2}$$



$F(v_i) = v_i^2$

$f(v_i) = 2v_i$

$$\phi(v_i) = v_i - \frac{1 - v_i^2}{2v_i} = 0 \Rightarrow v_i = ?\ \sqrt{1/3}$$

0.8 ϕ $0.8 - \frac{1-0.16}{1.6} = 0.5775$

$0.4 \Rightarrow$ VCG $\Rightarrow$ $w' = 1$, $p' = 0$ $\Rightarrow$ $w = 1$, $p = \phi^{-1}(0)$

0.3

n, $F(v_i)$, $f(v_i)$

$\phi(v_i) = 0$

$p = ? \rightarrow m\,?$

$Rev = p \times m$

v_1

v_2

v $\vdots$

v_k

$\vdots$

v_n

$m = ?$

$Rev = v_{m+1} \cdot m$

Profit Maximization in Mechanism Design

> Optimal Auction Design. Myerson 1981
>
> ch 13.2

Vickrey auction with reservation price r

单物品：VCG = second price auction

- Two biders
- $b_1, b_2 \in [0, 1]$ Random/uniform

- $min(b_1, b_2)$
- $Pr(min(b_1, b_2) > x) = (1 - x)^2$
- $F(x) = 1 - (1 - x)^2$
- $f(x) = F'(x) = 2 - 2x$

- $E(min(b_1, b_2)) = \int_0^1 xf(x)dx = \int_0^1 x(2 - 2x) = x^2 - 2/3x^3|_0^1 = 1/3$

Reserve Price

- $VCG_r : max(b_1, b_2) \geq r$ -> sell
- $payment = max(r, min(b_1, b_2))$
- $P(pay\ r) = r * (1 - r) * 2$
- $E(payment) = r * r * (1 - r) * 2 + x^2 - 2/3x^3|_r^1 = r^2 - 4/3r^3 + 1/3$

明显好于VCG 且1/2最优

Virtual Valuation

$\phi_i(v_i) = v_i - \frac{1 - F(v_i)}{f(v_i)}$

1. Collect all bids b_i
2. Compute $\phi_i(b_i)$
3. Apply VCG on $\phi_i(b_i)$, get allocation x_i, payment P_i
4. Final allocation is x_i, final payment is $\phi_i^{-1}(P_i)$

$v_i \in [0, 1)$

$\phi_i(b_i) = b_i - (1 - b_i) = 2b_i - 1 \in [-1, 1]$

$2b_i - 1 = 0$

$b_i = 1/2$

卖软件：保留价vcg-> fixed price auction

$\boldsymbol{\phi_i(v_i)}$单调非减能保证truthful