CS243: Introduction to Algorithmic Game Theory

Cake Cutting (Dengji ZHAO)

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Recap: Coalitional/Cooperative Game

- A set of agents N.
- Each subset of agents (coalition) $S \subseteq N$ cooperate together can generate some value $v(S) \in \mathbb{R}$. Assume $v(\emptyset) = 0$. N is called grand coalition. $v : 2^N \to \mathbb{R}$ is called the characteristic function of the game.
- The possible outcomes of the game is defined by $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$.

Recap: Core and Shapley Value

Definition (Core)

The core of the coalitional game (N, v) is a set of vectors $x \in \mathbb{R}^N$ such that x is efficient and $\forall_{S \subseteq N} \sum_{i \in S} x_i \geq v(S)$.

Definition (Shapley Value)

Given a coalitional game (N, v), the Shapley value of each player i is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S))$$

Recap: Cost Sharing

Definition

A cost sharing game (N, c) is defined by

- a set of n agents N.
- a cost function $c : 2^N \to \mathbb{R}$ and assume $c(\emptyset) = 0$.

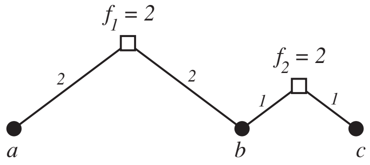

Figure 15.1. An example of the facility location game.

- $c(\{a\}) = 4, c(\{b\}) = 3, c(\{c\}) = 3$
- $c(\{a, b\}) = 6, c(\{b, c\}) = 4, c(\{a, c\}) = 7, c(\{a, b, c\}) = 8$

Cake Cutting

Cake Cutting

Cardinal Preferences

- A divisible resource C, say a cake.
- A set of n players to share/divide.
- Each player has valuation function v_i, which gives a value for each subset of C. We assume v_i is additive.

Question

How to divide the resource fairly?

Fairness

Proportionality Each player receives a piece that he values as at least $1/n$ of the value of the entire cake.

Envy-freeness Each player receives a piece that he values at least as much as every other piece.

Question: Does envy-freeness implies proportionality?

A Cake Cutting Procedure: Divide and Choose

- Two person share one cake.
- One person (the cutter) cuts the cake into two pieces.
- The other person chooses one (the chooser).

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Question

What is the best strategy for the cutter?
Does it satisfy *proportionality*?
Does it satisfy *envy-freeness*?

Proportional Cake Cutting: Last Diminisher

Question

How to extend Divide and Choose to more than two person settings?

- The players being ranged A, B, C, ... N.
- A cuts from the cake an arbitrary part.
- B has now the right, but is not obliged, to diminish the slice cut off.
- Whatever B does, C has the right (without obligation) to diminish still the already diminished (or not diminished) slice, and so on up to N.
- The rule obliges the "last diminisher" to take as his part the slice he was the last to touch.

Proportional Cake Cutting: Last Diminisher

Question

- Does Last Diminisher satisfy *proportionality*?
- Does Last Diminisher satisfy *envy-freeness*?

Proportional Cake Cutting: Moving-knife Protocol

(Proposed by Lester Dubins and Edwin Spanier in 1961.)

- The cake: interval [0,1].
- n players 1, 2, ..., n and a refree.

Moving-knife Protocol:

- Referee starts a knife at 0 and moves the knife to the right.
- Repeat: When the piece to the left of the knife is worth $1/n$ to a player, the player shouts "stop", receives the piece, and exits.
- When only one player remains, she gets the remaining piece.

Complexity of moving-knife protocol: $\Theta(n^2)$

Proportional Cake Cutting: Moving-knife Protocol

Question

- Does Moving-knife protocol satisfy *proportionality*?
- Does Moving-knife protocol satisfy *envy-freeness*?

Proportional Cake Cutting: Even Paz

(Proposed by S. Even and A. Paz, in 1963.)

Input:

- A piece of cake $[x, y]$.
- n agents. (Assume $n = 2^k$ for simplicity)

Recursive procedure:

- If $n = 1$, give $[x, y]$ to the single agent.
- Otherwise:
    - Each agent mark a point z such that $v([x, z]) = v([z, y])$.
    - Let z^* be the (n/2)-th mark from the left.
    - Recurse on $[x, z^*]$ with the left n/2 agents, and on $[z^*, y]$ with the right n/2 agents.

Proportional Cake Cutting: Even Paz

- Even Paz protocol uses a divide-and-conquer strategy, it is possible to achieve a division in time $O(n \log n)$.

Theorem

The Even Paz protocol produces a proportional allocation.

Theorem

Any protocol returning a proportional allocation needs $\Omega(n \log n)$ queries. [Edmonds and Pruhs, 2006]

Envy-free Cake Cutting

A query: either asks an agent her value of some piece, or asks her to cut a piece that her valuation is some value.

- $n = 2$ agents: 2 queries (Divide and Choose).
- $n = 3$ agents: 14 queries (Selfridge and Conway, 1960).
- $n = 4$ agents: 171 queries (Amanatidis et al., 2018).

Theorem

Any protocol for finding an envy-free allocation requires $\Omega(n^2)$ queries.

Envy-free Cake Cutting: Selfridge Conway procedure

Question

How to get an envy-free allocation among 3 players?

Stage 1:

- P_1 divides the cake into three pieces he considers of equal size.
- Let A be the largest piece according to P_2.
- P_2 trims A into A1 such that it has the same size as the second largest. Let the trimming piece be A2.
    - If P_2 thinks that the two largest parts are equal, then players chooses a part in order: P_3, P_2, P_1.

Envy-free Cake Cutting: Selfridge Conway procedure

Stage 2:

- P_3 chooses a piece among A1 and the two other pieces.
- P_2 chooses a piece with the limitation that if P_3 didn't choose A1, he must choose it.
- P_1 chooses the last piece leaving just the trimmings A2 to be divided.
  - A1 has been chosen by either P_2 or P_3, let the player who chose it P_A and the other player P_B.

Stage 3:

- P_B cuts A2 into three equal pieces.
- Each of players choose one piece in order: P_A, P_1, P_B.

Advanced Reading

- AGT Chapter 10.2
- *Computational Social Choice* by F. Brandt, V. Conitzer and U. Endriss