# CS243: Introduction to Algorithmic Game Theory

Cooperative Games and Cost Sharing (Dengji ZHAO)

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# Coalitional/Cooperative Game

- A set of agents $N$.
- Each subset of agents (coalition) $S \subseteq N$ cooperate together can generate some value $v(S) \in \mathbb{R}$. Assume $v(\emptyset) = 0$. $N$ is called grand coalition. $v : 2^N \rightarrow \mathbb{R}$ is called the characteristic function of the game. $v$ is often assumed to be monotonic: $S \subseteq T \Rightarrow v(S) \leq v(T)$.
- The possible outcomes of the game is defined by $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$.

# Example

- Three agents $\{1, 2, 3\}$.
- $v(\{1\}) = v(\{2\}) = v(\{3\}) = 1$;
  $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 2$; $v(\{1, 2, 3\}) = 3$.

# Core

**Definition**

For the grand coalition $N$, the allocation vector $x \in \mathbb{R}^N$ satisfy:

Efficiency if $\sum_{i \in N} x_i = v(N)$.

Individual Rationality if $\forall_{i \in N} x_i \geq v(\{i\})$.

**Definition (Core)**

The core of the coalitional game $(N, v)$ is a set of vectors $x \in \mathbb{R}^N$ such that $x$ is efficient and $\forall_{S \subseteq N} \sum_{i \in S} x_i \geq v(S)$.

# Shapley Value: a Fair Distribution of Payoffs

Given a coalitional game $(N, v)$, the Shapley value of each player $i$ is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S))$$

2012 Nobel Memorial Prize in Economic Sciences

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Calculate the Shapley value for the following game:

- Three agents $\{1, 2, 3\}$.
- $v(S) = 1$ if $S \in \{\{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, otherwise $v(S) = 0$.
- $\phi_1(v) = \phi_2(v) = \frac{1}{6}$ and $\phi_3(v) = \frac{2}{3}$.

# Properties of Shapley Value

- **Efficiency**: $\sum_{i \in N} \phi_i(v) = v(N)$.
- **Symmetry**: If $i$ and $j$ are two players who are equivalent in the sense that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N$ s.t. $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.
- **Linearity**: $\phi_i(v + w) = \phi_i(v) + \phi_i(w)$.
- **Zero player** (null player): $\phi_i(v) = 0$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$.

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**Question**

Is the Shapley value in the core? [advanced reading]

# Cost Sharing

In the above coalitional game $(N, v)$, we assumed that $v(S) \geq 0$, it is possible that $v(S) \leq 0$ (which becomes a cost sharing game).

**Definition**

A cost sharing game $(N, c)$ is defined by

- a set of $n$ agents $N$.
- a cost function $c : 2^N \to \mathbb{R}_+$ and assume $c(\emptyset) = 0$.

# Cost Sharing

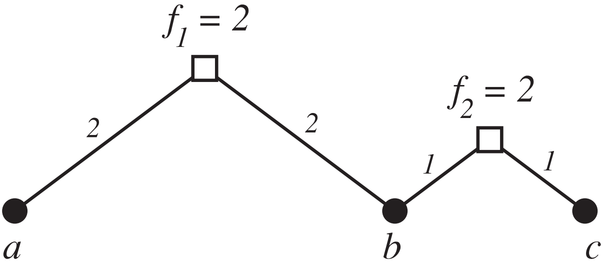



**Figure 15.1.** An example of the facility location game.

- $c(\{a\}) = 4, c(\{b\}) = 3, c(\{c\}) = 3$
- $c(\{a,b\}) = 6, c(\{b,c\}) = 4, c(\{a,c\}) = 7, c(\{a,b,c\}) = 8$

# Core of Cost Sharing

**Definition (Core)**

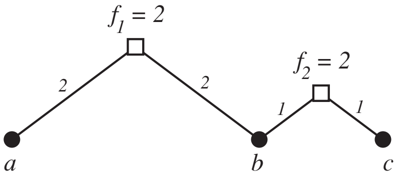
A vector $\alpha \in \mathbb{R}^N$ is in the core of a cost sharing game $(N, c)$ if

- $\sum_{i \in N} \alpha_i = c(N)$
- $\forall_{S \subseteq N} \sum_{j \in S} \alpha_j \leq c(S)$

# Core of Cost Sharing

Questions:

- Is $(4, 2, 2)$ in the core of the following game?
- Is $(4, 1, 3)$ in the core of the following game?

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# Advanced Reading

- AGT Chapter 15: *Cost Sharing*.