

# CS243: Introduction to Algorithmic Game Theory

Matching (Dengji ZHAO)

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# Matching

## Matching (Mechanism Design without Money)

- Agents in two sides.
- A **matching**: each agent is assigned to at most one agent on the other side.

### *Two-sided matching*

- Agent in one set has strict preferences over agents in other set, e.g. students to schools

### *One-sided matching*

- Only one side has strict preference on the other side, e.g. house allocation

# One-sided Matching: House Allocation

- Without initial allocation
- With initial allocation

# House Allocation without Initial Allocation

- A set of agents, a set of items/houses
- Each agent has a strict preference on houses

# House Allocation without Initial Allocation

- A set of agents, a set of items/houses
- Each agent has a strict preference on houses

## Questions

- Is it possible to design a truthful mechanism under this setting?
- Does Gibbard-Satterthwaite theorem hold here?

# House Allocation without Initial Allocation

- A set of agents, a set of items/houses
- Each agent has a strict preference on houses

## (Randomized) Serial dictatorship:

- Predefine a ranking of agents (could be randomized).
- Let each agent from the top of the ranking to choose her preferred house first.

# Pareto Optimality

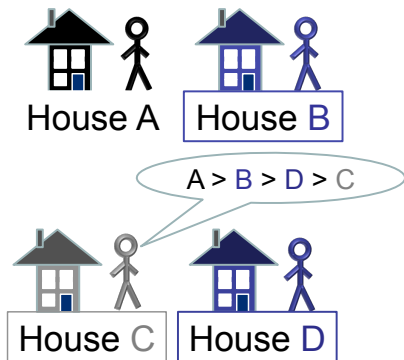
## Definition

**Pareto optimality** is a state of allocation of resources from which it is impossible to reallocate so as to make any one individual or preference criterion better off without making at least one individual or preference criterion worse off.

- Question: is serial dictatorship Pareto optimal?

# House Allocation/Exchange with Initial Allocation

- A set of agents, each agent has one house
- Each agent also has a strict preference on all houses





# Blocking Coalition

## Definition

A **blocking coalition** can, by trading/exchange among themselves, receive homes that each strictly prefers (or is equivalent) to the home she receives under current allocation, with at least one agent being strictly better off.

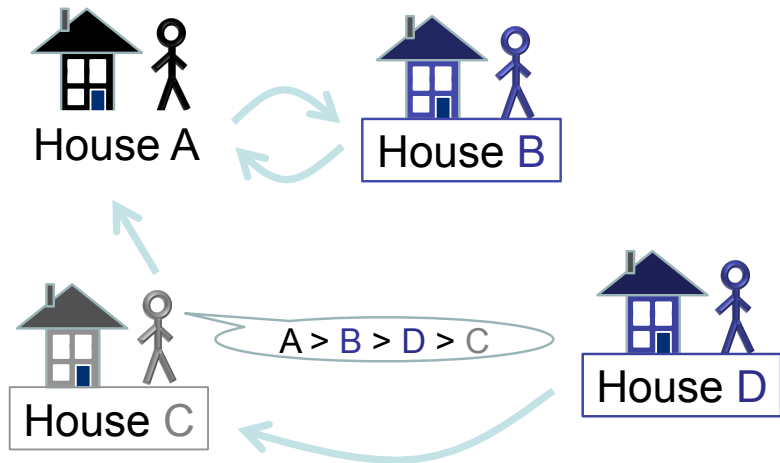
## Question

If an allocation is blocked by a subset of the agents, how to change the allocation to remove all blocking coalitions?

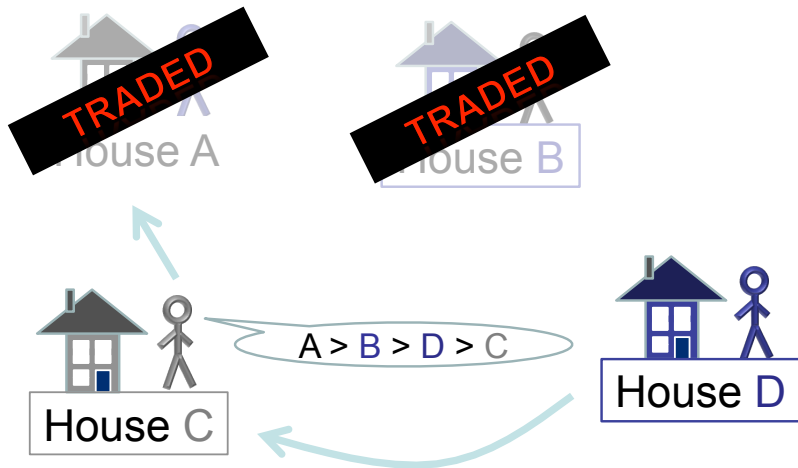
# Top Trading Cycle Mechanism

- 1 Each agent points to most preferred house (allow self-edge).
- 2 Trade on cycles, agents and houses leave market.
- 3 Each remaining agent points to its most preferred, remaining house.
- 4 Repeat (#1, #2, #3), until no agents left.
  - AGT, Definition 10.5 is wrong!

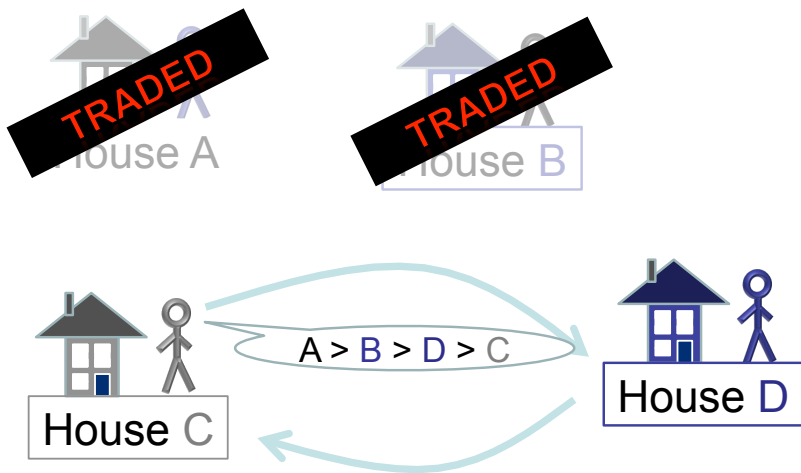
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## Questions:

- Is TTC truthful?
- Is TTC pareto optimal?

# Two-sided Matching

- Examples
  - Students to schools
  - Doctors to hospitals
  - TAs to professors
  - Boys to girls
- Agents in two sets; agent in one set has strict preferences over agents in the other set
- A **matching**: each agent assigned to at most one agent on the other side

# Stable Matching

## Definition

A **stable matching** is a matching with no **blocking pair**, a blocking pair is two agents who prefer to match with each other.



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How to reach a stable matching?

## Boy-Proposing Deferred Acceptance

- 1 Each man proposes to his top-ranked choice.
- 2 Each woman who has received at least two proposals keeps (tentatively) her top-ranked proposal and rejects the rest.
- 3 Each man who has been rejected proposes to his top-ranked choice among the women who have not rejected him.
- 4 Each woman who has at least two proposals (including ones from previous rounds) keeps her top-ranked proposal and rejects the rest.

# Boy-Proposing Deferred Acceptance

## Theorem

*Boy-Proposing DA generates a stable matching*

- Suppose  $(b,g)$ ,  $(b',g')$  in the matching
- For contradiction, suppose  $(b,g')$  is a blocking pair
- $\Rightarrow$  boy  $b$  prefers  $g'$  to  $g$ , and proposes to  $g'$  before girl  $g$
- $\Rightarrow$  girl  $g'$  prefers boy  $b'$  to boy  $b$  (since match improves for girls during the DA)
- $\Rightarrow (b,g')$  is not a blocking pair. Contradiction.

# Is Stable Matching Unique?

- What will happen with **Girl-proposing DA**?

# Advanced Reading

- AGT, Chapter 10: Mechanism Design without Money